Relationship Between Modulus and Density for High-Density Closed-Cell Thermoplastic Foams

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Synopsis

It is necessary to establish a relationship between Young's modulus and density for a closed-cell plastic foam in order to predict the flexural stiffness properties of integral skinned foamed thermoplastics. It has been shown experimentally that there exists a square-power relationship between modulus and density for high-density closed-cell thermoplastic foams, but this has not been confirmed by existing theoretical models for foams. These models are reviewed and finite element analyses of "circular hole in square plate" and "spherical hole in solid cube" models are presented which give close agreement with the empirical results.

INTRODUCTION

Integral skinned foamed thermoplastics called "structural foams" and "sandwich moldings" are widely used because they provide good bending stiffness properties for low weight and cost. In order to predict their flexural stiffness properties, it is necessary to establish a relationship between modulus and density for their foamed core. Their morphology is more complicated than most constructed sandwich materials because densities are not uniform within a molding but vary from a value of nearly unity relative to the density of solid material near the outside of the molding to around 0.3 in the center layers. It is necessary, therefore, to treat the core either in some average way or to take account of the varying density and modulus. In both cases a basic relationship between modulus and density is required.

It is not easy to establish reliable experimental values for foam modulus, because of the complex nature of these materials and the effect of the molding process on the properties. Two independent sources[1,2] give empirical relationships between modulus and density for a high-density foamed thermoplastic which approximate very closely to a square-power relationship:

$$E_f/E_s = (\rho_f/\rho_s)^2$$

where E_f and E_s are the Young's moduli of foamed and solid materials, respectively; and ρ_f and ρ_s are the respective densities.

Several theoretical models have been developed to describe the mechanical behavior of foams, but they do not predict accurately the empirical relationship given above. These models are reviewed in this paper and a finite element analysis by the authors is described which gives values close to the experimental results.

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REVIEW OF THEORETICAL MODELS

Law of Mixtures Model

This model assumes that the modulus of the foamed material is in direct proportion to the moduli of the component materials (solid thermoplastic and air) and their relative content by volume. Thus,

$$E_f = nE_a + (1 - n)E_s$$

where f, s, and a represent the foamed plastic, solid plastic, and air, respectively; and n is the volume fraction of air ($0 < n < 1$). Taking $E_a = 0$ and $n = 1 - \rho_f/\rho_s$ then

$$\frac{E_f}{E_s} = 1 - n = \frac{\rho_f}{\rho_s}$$

This relationship is shown in Figure 3.

Strength of Materials Models

These models assume uniform elastic deformation of an idealized element of a homogeneous foam structure.

Square in Square

A repeated element is assumed to comprise a square hole (side length h) cut out of a square plate (side length H) of unit thickness, as shown in Figure 1. An applied axial load F is assumed to be uniformly distributed and to cause a uniform linear elastic deformation e (i.e., stress concentration effects are ignored). Applying Hooke's law to the whole element,

$$e = \frac{F}{H}\frac{H}{E_f}$$

in terms of solid material alone,

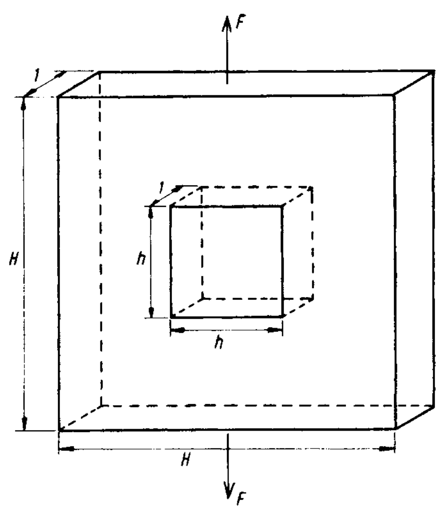

Fig. 1. Square-in-square model.

$$e = \frac{F}{E_s}\left[\frac{H-h}{h} + \frac{h}{H-h}\right]$$

Eliminating F/e gives

$$\frac{E_f}{E_s} = \frac{1 - (h/H)}{1 - (h/H) + (h^2/H^2)}$$

But h^2/H^2 = volume fraction of void, n $(= 1 - \rho_f/\rho_s)$. Therefore,

$$\frac{E_f}{E_s} = \frac{1 - n^{1/2}}{1 - n^{1/2} + n}$$

Cube in Cube

This is a logical extension to the model above. It considers the uniform uniaxial deformation of a cubic element (side length H) containing a central cubic void (side length h). A similar analysis gives the following relationship:

$$\frac{E_f}{E_s} = \frac{1 - n^{2/3}}{1 - n^{2/3} + n} \qquad \text{where} \qquad n = \frac{h^3}{H^3} = 1 - \frac{\rho_f}{\rho_s}$$

Modified Cube in Cube

Two "cube in cube" elements adjacent in the direction of loading are shown in Figure 2. If the foamed plastic comprises sets of "cube in cube" elements in series, it is not unreasonable to assume that the sections between the voids (shown hatched in Fig. 2) are not load bearing. If such regions are ignored, then

$$\frac{E_f}{E_s} = \frac{H^2 - h^2}{H^2} = 1 - n^{2/3}$$

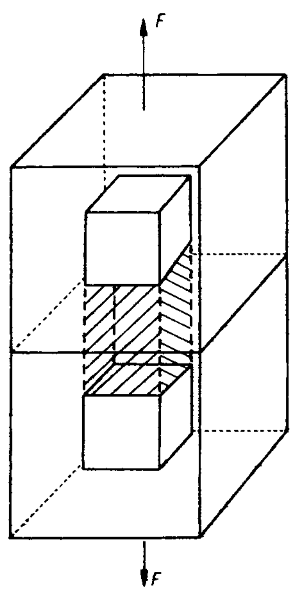

Fig. 2. Modified cube-in-cube model.

Mackenzie and Kerner Models

These models comprise a spherical void (Mackenzie[3]) or nonreinforcing grain (Kerner[4]) encapsulated in a spherical shell of matrix material which itself is encapsulated in a spherical shell of a homogeneous material having average properties. An elasticity analysis for hydrostatic loading was performed to give values of bulk and shear modulus.

Kerner's analysis, when interpreted for an air/solid system, gives the following expressions for shear modulus G and bulk modulus K:

$$G_f = \frac{G_s(1-n)/15(1-\nu_s)}{n/(7-5\nu_s) + (1-n)/15(1-\nu_s)}$$

$$K_f = \frac{K_a n/(3K_a + 4G_s) + K_s(1-n)/(3K_s + 4G_s)}{n(3K_a + 4G_s) + (1-n)/(3K_s + 4G_s)}$$

where ν_s is Poisson's ratio for the solid material.

Baxter and Jones[5] have combined the above equations with the relations between elastic constants for an isotropic material,

$$E = 3K(1-2\nu) = 2G(1+\nu)$$

to produce a relationship between Young's modulus and volume fraction of air n. This is plotted in the form E_f/E_s against ρ_f/ρ_s in Figure 4.

Rusch[6] derived the same result independently; he also manipulated Mackenzie's analysis to give a similar relationship (shown in Fig. 4), having made subsidiary assumptions to prevent physically meaningless results.

Lederman Model

Lederman[7] developed a theory by Gent and Thomas[8] which assumed the foam to comprise intersecting thin threads forming a lattice structure with nondeforming globules at the junction points.

Stress analysis of this model gives the following expression for modulus ratio when $\nu_f = 0.25$:

$$\frac{E_f}{E_s} = \frac{B^2}{1+B} N$$

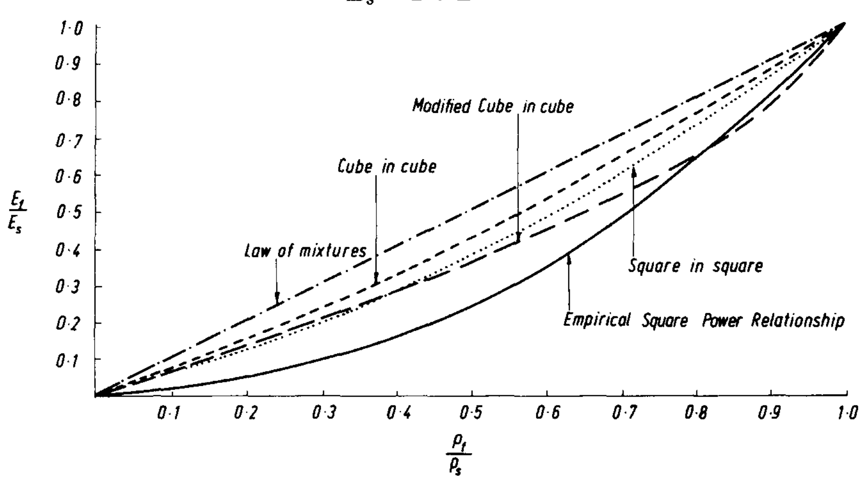

Fig. 3. Comparison of theoretical models.

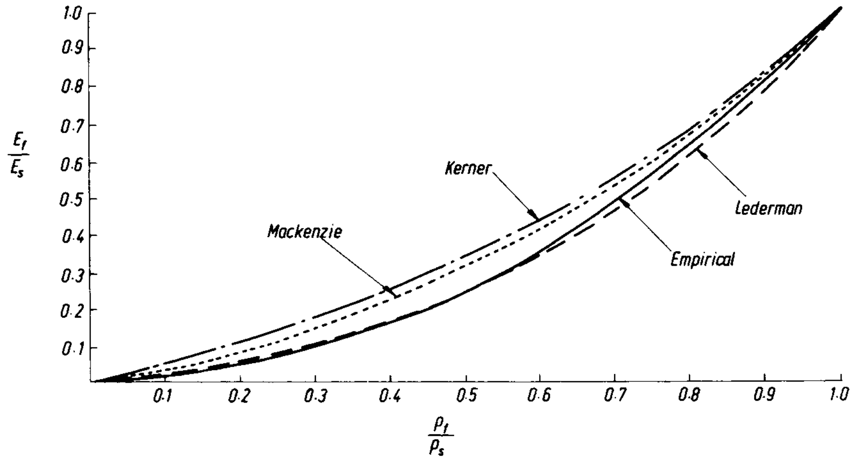

Fig. 4. Comparison of theoretical models.

where N is a structural parameter (equals unity for an isotropic material) and B is a geometric parameter related to N and the density ratio ρ_f/ρ_s. This theory was specifically intended for low-density open-cell foams, and direct calculation of B for high densities gives values of E_f/E_s greater than unity. It is possible (though perhaps questionable) to apply the condition that $E_f/E_s = 1$ when $\rho_f/\rho_s = 1$. This gives a limiting value of B (equal to 2.732 for N equal to unity). Values of B for other densities can then be interpolated graphically ignoring the mathematical relationship between B and density ratio for values of the latter above 0.5. The modulus/density relationship shown in Figure 4 was plotted on this basis.

Mehta–Columbo Model

This analysis[9] is based on a model by Halpin and Tsai[10] for anisotropic composite materials. For a composite containing solid polymer and air, Mehta and Columbo derived the relationship

$$\frac{E_f}{E_s} = \frac{\rho_f/\rho_s}{1 + [1 - (\rho_f/\rho_s)]/z}$$

where z is a constant related to cell size and orientation. For spherical air cells, z equals 0.5. The modulus/density relationship for this case is shown in Figure 5. Also shown are the corresponding relationships with z having values 0.4 and 0.6. These show the extent to which the model is influenced by this factor.

COMPARISON OF MODELS

All the models described contain simplifying assumptions of structure and homogeneity that are clearly not borne out in practice. Their validity must be judged, at least partially, in terms of how closely they predict the experimental square-power relationship between relative modulus and density.

As shown in Figure 3, the "law of mixtures" and "strength of materials" approaches are clearly inadequate. The Mackenzie and Kerner models (Fig. 4)

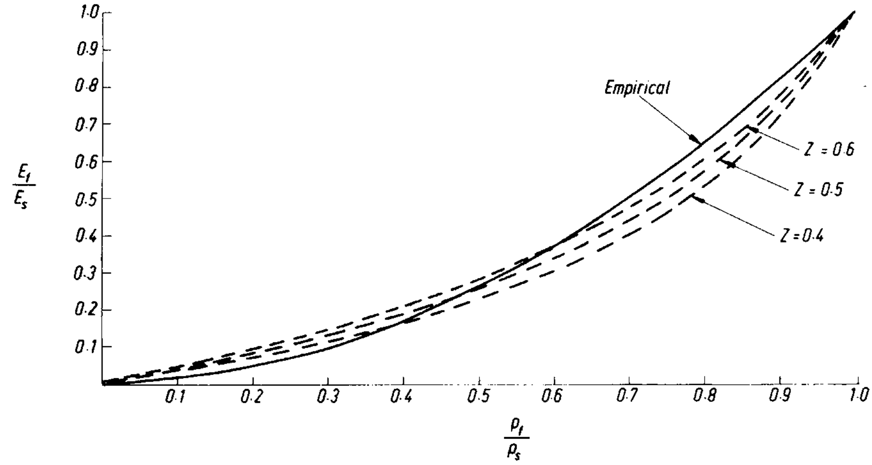

Fig. 5. Mehta–Columbo model.

are better, but they still overestimate the stiffness of foamed material and are in error by about 40% at a density ratio of 0.5.

Lederman's model can be interpreted to give an excellent prediction of the empirical curve, as shown in Figure 4. Its validity though must be in question because it assumes an open-cell structure which is manifestly untrue for high relative densities. Also, the geometric parameter has to be adjusted in order to prevent meaningless results at these high densities.

The Mehta–Columbo model gives reasonable agreement with practice. The predicted modulus-versus-density relationship has however a different shape to the empirical one (whatever value of the adjustable structural parameter is chosen) so that errors at either high or low densities are inevitable.

FINITE ELEMENT APPROACH

The finite element technique[11] provides a numerical method of stress analysis. An approximate solution to a continuum problem with a complex stress distribution is obtained by idealizing the structure was an assembly of interconnected elements. Within each element a simplified displacement distribution is assumed. In the analysis used by the authors, plane triangular elements having a linear displacement distribution (and therefore constant strain and stress) were used. With the finite element method it is possible to analyze models similar to the "strength of materials" models described earlier, without having to assume a greatly simplified stress distribution.

It is assumed, in company with most other models of foamed plastics, that the foam structure is uniform and homogeneous. Although this is not implicit in the finite-element method, the solid-material phase was assumed to be isotropic and linearly elastic. Also, because of limitations in the computer programs available to the authors, only two-dimensional models were analyzed, though it proved possible to extend these to three dimensions as described later.

The finite-element meshes shown in Figures 6 and 11 represent the smallest repeating unit in a model foam structure for which the boundary conditions can be easily defined. For both these meshes the straight portions of all four boundaries must, after deformation due to uniaxial loading, remain straight and parallel to their initial directions.

The mesh shown in Figure 6 enabled a "circular hole in square plate" model of unit thickness to be analyzed in plane stress. The element configuration and numbering allowed the hole to be increased in size by the removal of rings of elements.

A uniform displacement was applied normal to the upper boundary of the mesh, and all four straight boundaries were thereafter constrained to move only in their initial directions. Constraining the side boundaries in this way introduces biaxial loading, but tests showed that it made negligible difference to the ratio of stiffness between foamed and solid materials, providing the same boundary conditions were applied when considering the solid material alone. The resultant stresses along the displaced edge were plotted for different hole sizes, as shown in Figure 7. The area under each of these curves gives the total force required to uniformly displace the upper boundary. This is proportional to the stiffness of the "model," which in turn is proportional to the modulus of the foamed material E_f. The stiffness of an identically dimensioned cuboid of solid material with the same boundary conditions was calculated to be 3530 N/mm. Hence, ratios equivalent to modulus ratios were obtained. The density ratios ρ_f/ρ_s were calculated by simple geometry. As shown in Figure 8, this gives a good approximation to the empirical square-power relationship despite the assumptions made.

It is possible to simulate a three-dimensional spherical void in solid-cube model using the two-dimensional mesh described above. The mesh was reanalyzed as before but under conditions of plane strain. Hence, a graphic relationship between hole size and force required for uniform axial displacement was obtained.

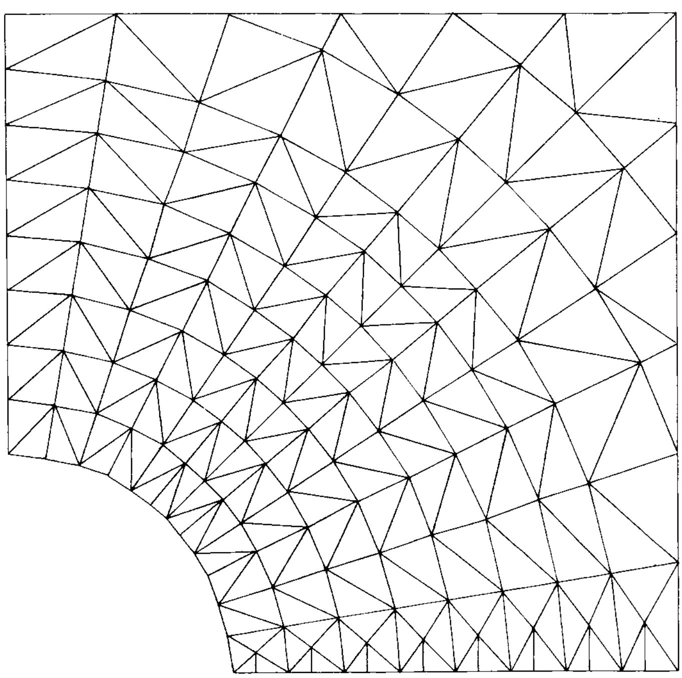

Fig. 6. Circular hole in square plate mesh.

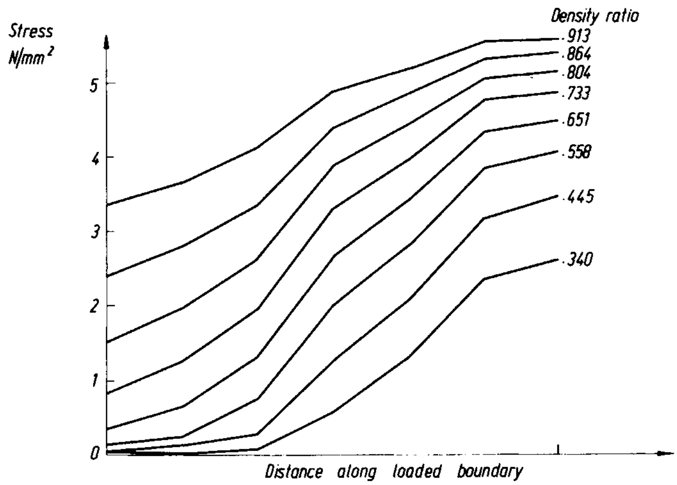

Fig. 7. Stress distribution along displaced boundary.

A sphere in a cube can be approximated by a series of thin (two-dimensional) square slices containing circular holes of average diameter, as shown in Figure 9. Neglecting interactions between these slices, a set of plane "circular hole in square plate" models is equivalent to a "spherical hole in cube" model which is uniaxially deformed, with its four sides constrained to move only in their initial planes. Summation of the forces for each slice and knowledge of the force required to displace the solid equivalent yields the modulus ratio E_f/E_s. The corresponding density ratio is determined geometrically. The quality of this model is demonstrated in Figure 10. The predicted results are all within 7% of the empirical square power relationship.

The fact that both the two- and three-dimensional analyses predict higher foam stiffnesses than exist in practice at relative densities less than around 0.6 can be explained in terms of the observed morphology of thermoplastic foams.[12] It is not unreasonable to assume that at higher densities the cells, which are nearly

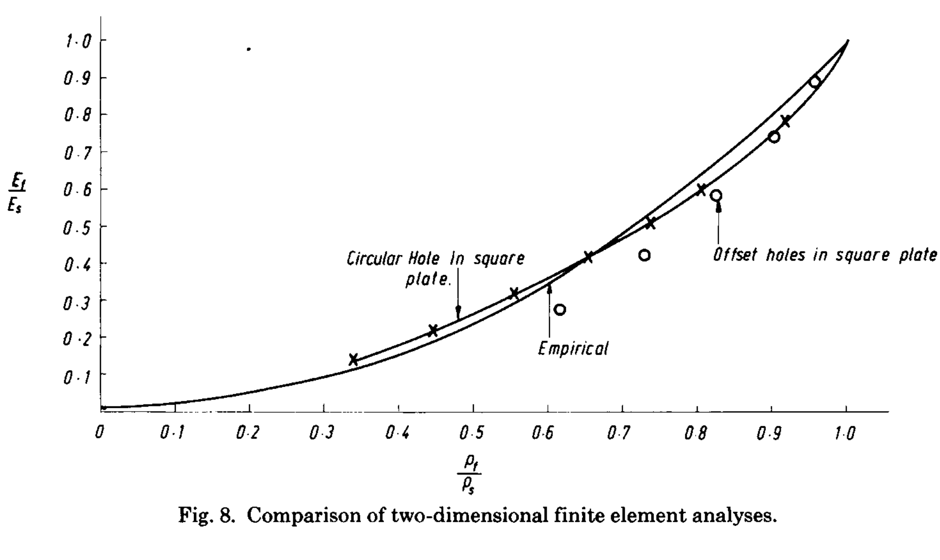

Fig. 8. Comparison of two-dimensional finite element analyses.

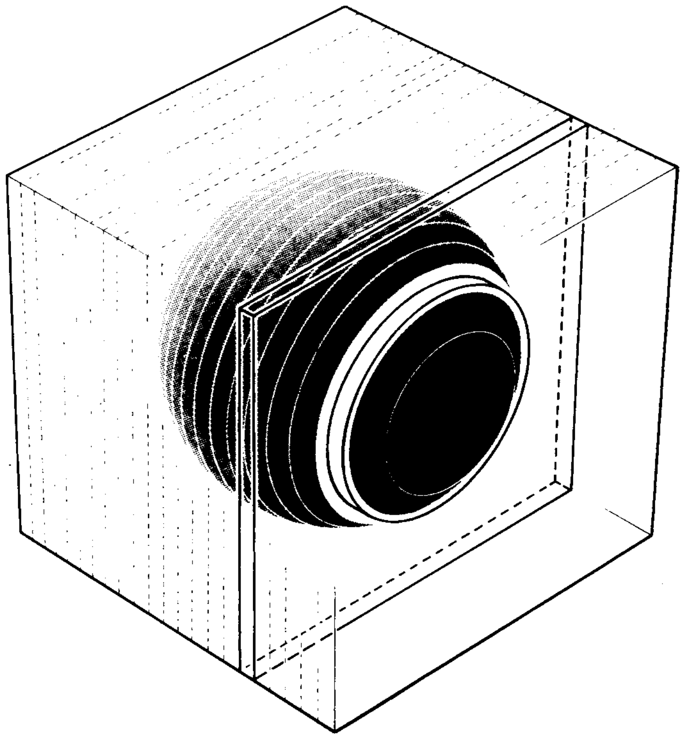

Fig. 9. Approximate sphere in cube model showing "two-dimensional slice."

randomly centered, do not interact and can be modeled reasonably by a "sphere in cube" model. At lower densities there is a tendency for foams to form a three-dimensional network with some symmetry. The structure given by the "sphere in cube" model (an array of spherical holes with interleaved blocks of solid material continuous in the loading direction) does not exist. This effect was briefly investigated by a two-dimensional (plane stress) analysis of an "offset circular holes in square plate" model, the mesh for which is shown in Figure 11.

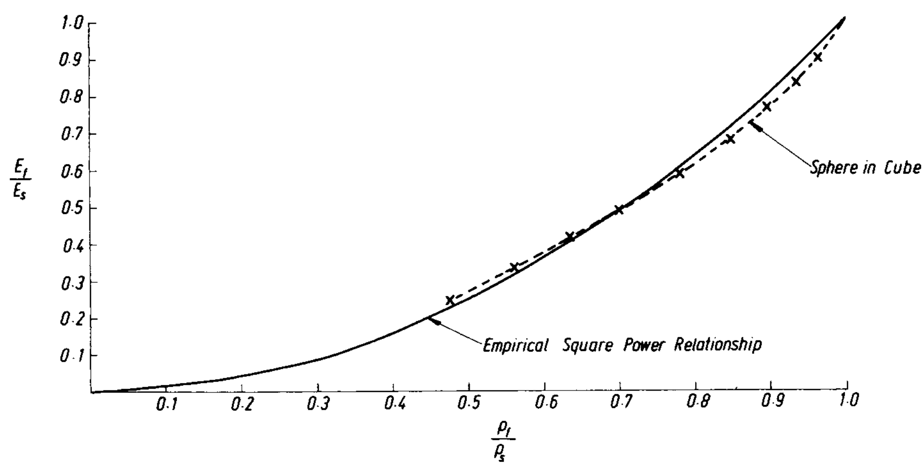

Fig. 10. Three-dimensional analysis.

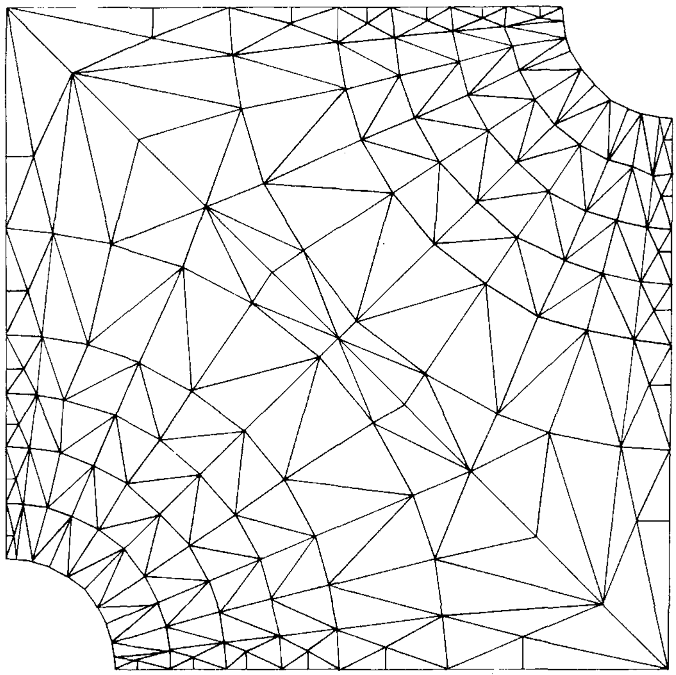

Fig. 11. Offset holes in square plate mesh.

The predicted stiffnesses (shown in Fig. 9) were less than both those predicted by the "circular hole in plate" model and empirical values because the "offset holes" model contains less area of continuous solid material. In order to study low foam densities, a "cubic hole in solid cube" model was constructed from plane strain "square hole in square plate" slices using two-dimensional finite element analyses and the method described above for the "sphere in cube" model. This gave higher stiffness ratios than did the "sphere in cube" model since, for the same density, a "cube in cube" model has a greater volume of material continuous in the direction of loading.

CONCLUSIONS

Finite element analysis of a model with some physical resemblance to a real high-density closed-cell foam structure predicts a relationship between the relative modulus and relative density of foamed and solid materials similar to that measured experimentally.

The deficiencies at low relative densities can be explained in terms of the known morphology of thermoplastic foams.

The finite element approach gives much better agreement with practice than most previous theoretical models. It is interesting to note that the relative modulus-versus-density relationship obtained from the two-dimensional "circular hole in square plate" finite element analysis corresponds almost exactly with that from the Mehta–Columbo model using the value 0.6 for the structural parameter z.

The Lederman model gives excellent results but is of doubtful validity for high densities and is based on an open cell structure. For low densities, however, the thin "straight" walled structures of both open and closed cell foams are broadly similar, so that Lederman's model may have some physical validity for low-density closed-cell foams.

In practice, no theoretical model can be really valid because of the wide possible variations in cell size, shape, and orientation caused by the molding process. However, for predictive purposes it is necessary to use some generalized simple analysis, and experimental evidence[1,2,8] has shown that the square-power relationship between relative modulus and relative density can be used with some confidence not only for high-density rigid closed cell foams but also for foams that are flexible, open cell, or of low density.

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